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Spectrum generating algebras for position-dependent mass oscillator Schrödinger equations

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Abstract

The interest of quadratic algebras for position-dependent mass Schrödinger equations is highlighted by constructing spectrum generating algebras for a class of d-dimensional radial harmonic oscillators with $d \geqslant 2$ and a specific mass choice depending on some positive parameter α. Via some minor changes, the one-dimensional oscillator on the line with the same kind of mass is included in this class. The existence of a single unitary irreducible representation belonging to the positive-discrete series type for $d \geqslant 2$ and of two of them for $d = 1$ is proved. The transition to the constant-mass limit $\alpha \to 0$ is studied and deformed su(1,1) generators are constructed. These operators are finally used to generate all the bound-state wavefunctions by an algebraic procedure.

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1. Introduction

During recent years, quantum mechanical systems with a position-dependent (effective) mass (PDM) have attracted a lot of attention and inspired intense research activities. They are indeed very useful in the study of many physical systems, such as electronic properties of semiconductors [1] and quantum dots [2], nuclei [3], quantum liquids [4], ^{3}He clusters [5], metal clusters [6], etc.

Furthermore, the PDM presence in quantum mechanical problems may reflect some other unconventional effects, such as a deformation of the canonical commutation relations or a curvature of the underlying space [7]. It has also recently been signalled in the rapidly growing field of PT-symmetric [8, 9] (or pseudo-Hermitian [10] or else quasi-Hermitian [11]) quantum mechanics as occurring in the Hermitian Hamiltonian equivalent to some PT-symmetric systems at lowest order of perturbation theory [12–14].

Looking for exact solutions of the Schrödinger equation with a PDM has become an interesting research topic because such solutions may provide a conceptual understanding of some physical phenomena, as well as a testing ground for some approximation schemes. The generation of PDM and potential pairs leading to exactly solvable, quasi-exactly solvable or conditionally exactly solvable equations has been achieved by extending some methods known in the constant-mass case, such as point canonical transformations [15–34], Lie algebraic methods [35–39] and supersymmetric quantum mechanical techniques (or related intertwining operator methods) [7, 18, 19, 21, 22, 24, 34, 35, 40–48].

Another powerful tool used in standard quantum mechanics is that of nonlinear algebras, more specifically quadratic ones. For one-dimensional systems allowing exact solutions, such algebras may help us to understand the relation between the time evolution of classical dynamical variables and that of corresponding quantum operators, while providing a general method for constructing spectrum generating algebras [49] (see also [50]). In more than one dimension, they are a clue to classifying superintegrable systems with integrals of motion quadratic in the momenta [51–53] and to solving the Schrödinger equation for such systems [54–56].

In a PDM context, there has been no systematic use of quadratic algebras so far, although the presence of one of them has been signalled in a one-dimensional problem [43]. To start filling in this gap, we have recently considered the quadratic algebra generated by the integrals of motion of a two-dimensional superintegrable PDM system and shown how a deformed parafermionic oscillator realization of this algebra allows one to derive the bound-state energy spectrum [57].

In the present paper, we turn ourselves to another aspect of quadratic algebras, namely their occurrence as spectrum generating algebras, which we shall illustrate with the simplest example, corresponding to a harmonic oscillator potential. For a constant mass, it is well known (see, e.g., [58]) that all the states of such a potential with a given parity in one dimension or with a given angular momentum l in more than one dimension belong to a single unitary irreducible representation of an su(1,1) Lie algebra. The corresponding lowest-energy state is annihilated by the lowering generator, while the remaining states can be obtained from it by repeated applications of the raising generator. We plan to show that for a specific PDM choice, similar results apply except that su(1,1) gets deformed. We shall establish that a quadratic algebra approach provides us with a key to constructing such a deformed algebra, while allowing us at the same time to derive the bound-state energy spectrum.

In section 2, we present the Schrödinger equation of a PDM d-dimensional radial harmonic oscillator ($d \geqslant 2$) and review its bound-state energy spectrum. The corresponding spectrum generating algebra is constructed in section 3. In section 4, we show how the general d-dimensional results can be applied to the one-dimensional oscillator on the full line with a similar PDM. Finally, section 5 contains the conclusion.

2. Schrödinger equation of a PDM *d*-dimensional radial harmonic oscillator

Whenever both the PDM $m(r)$ and the potential $V(r)$ only depend on the radial variable r, the corresponding d-dimensional Schrödinger equation is separable in spherical coordinates. On writing the radial wavefunction as $r^{-(d-1)/2}\psi(r)$, so that the normalization condition for $\psi(r)$ reads

$$\int_0^\infty |\psi(r)|^2 \mathrm{d}r = 1, \tag{2.1}$$

we end up with the radial equation

$$\left(-\frac{\mathrm{d}}{\mathrm{d}r}\frac{1}{M(r)}\frac{\mathrm{d}}{\mathrm{d}r} + \tilde{V}_{\text{eff}}(r)\right)\psi(r) = E\psi(r). \tag{2.2}$$

Here, $M(r)$ is the dimensionless form of the mass function $m(r) = m_0 M(r)$, we have taken units wherein $\hbar = 2m_0 = 1$ and

$$\tilde{V}_{\text{eff}}(r) = V_{\text{eff}}(r) - \frac{(d-1)M'}{2rM^2} + \frac{L(L+1)}{Mr^2},$$

where a prime denotes derivative with respect to r, L is defined by $L = l + (d-3)/2$ in terms of the angular momentum quantum number l and $V_{\text{eff}}(r)$ is the effective potential that would arise in the Cartesian coordinate approach to the problem (see equation (2.3) of [24]).

Let us now consider a PDM d-dimensional harmonic oscillator, whose radial Schrödinger equation is obtained by replacing in the constant-mass one the radial momentum $p_r = -\mathrm{i}\,\mathrm{d}/\mathrm{d}r$ by some deformed operator, $\pi_r = \sqrt{f(\alpha; r)}\, p_r \sqrt{f(\alpha; r)}$, where $f(\alpha; r) = 1 + \alpha r^2$ and α is a positive real constant. The result of this substitution reads

$$\left(\pi_r^2 + \frac{L(L+1)}{r^2} + \frac{1}{4}\omega^2 r^2\right)\psi^{(\alpha)}(r) = E^{(\alpha)}\psi^{(\alpha)}(r) \tag{2.3}$$

which is equivalent to (2.2) with

$$M(\alpha; r) = \frac{1}{f^2(\alpha; r)} = \frac{1}{(1+\alpha r^2)^2}$$

and

$$\tilde{V}_{\text{eff}}(r) = \frac{L(L+1)}{r^2} + \frac{1}{4}(\omega^2 - 8\alpha^2)r^2 - \alpha$$

or

$$V_{\text{eff}}(r) = \tfrac{1}{4}\{\omega^2 - 4\alpha^2[L(L+1) + 2d]\}r^2 - \alpha[2L(L+1) + 2d - 1].$$

Observe that the constant-mass limit corresponds to $\alpha \to 0$, in which case equation (2.3) gives back the standard constant-mass equation.

Supersymmetric quantum mechanical methods, combined with deformed shape invariance, have shown [47] that the PDM Schrödinger equation (2.3) has an infinite number of bound states giving rise to a quadratic energy spectrum

$$E_{n,L}^{(\alpha)} = \alpha\left(4n^2 + 4n(L+1) + L + 1 + (4n + 2L + 3)\frac{\lambda}{\alpha}\right), \qquad n = 0, 1, 2, \ldots, \tag{2.4}$$

where $\lambda = \frac{1}{2}(\alpha + \Delta)$ and $\Delta = \sqrt{\omega^2 + \alpha^2}$. In the same work, the lowest-energy wavefunction (for given L) has been obtained in the form

$$\psi_{0,L}^{(\alpha)}(r) = \mathcal{N}_{0,L}^{(\alpha)} r^{L+1} f^{-[\lambda + (L+2)\alpha]/(2\alpha)}, \tag{2.5}$$

where the normalization coefficient $\mathcal{N}_{0,L}^{(\alpha)}$ can be easily determined from (2.1) as

$$\mathcal{N}_{0,L}^{(\alpha)} = \left(\frac{2\alpha^{L+\frac{3}{2}}\Gamma\left(\frac{\lambda}{\alpha} + L + 2\right)}{\Gamma\left(L + \frac{3}{2}\right)\Gamma\left(\frac{\lambda}{\alpha} + \frac{1}{2}\right)}\right)^{1/2}.$$

Some lengthy calculations along the same lines also yield [59]

$$\psi_{n,L}^{(\alpha)}(r) = \frac{\mathcal{N}_{n,L}^{(\alpha)}}{\mathcal{N}_{0,L}^{(\alpha)}} P_n^{\left(\frac{\lambda}{\alpha} - \frac{1}{2}, L + \frac{1}{2}\right)}(t)\,\psi_{0,L}^{(\alpha)}(r), \tag{2.6}$$

where $P_n^{\left(\frac{\lambda}{\alpha} - \frac{1}{2}, L + \frac{1}{2}\right)}(t)$ is a Jacobi polynomial [60] in the variable

$$t = 1 - \frac{2}{f} = \frac{-1 + \alpha r^2}{1 + \alpha r^2} \tag{2.7}$$

and

$$\frac{\mathcal{N}_{n,L}^{(\alpha)}}{\mathcal{N}_{0,L}^{(\alpha)}} = \left(\frac{\Gamma\left(L+\frac{3}{2}\right)\Gamma\left(\frac{\lambda}{\alpha}+\frac{1}{2}\right) n! \left(\frac{\lambda}{\alpha}+2n+L+1\right)\Gamma\left(\frac{\lambda}{\alpha}+n+L+1\right)}{\Gamma\left(\frac{\lambda}{\alpha}+L+2\right)\Gamma\left(\frac{\lambda}{\alpha}+n+\frac{1}{2}\right)\Gamma\left(n+L+\frac{3}{2}\right)} \right)^{1/2}. \tag{2.8}$$

Since in the constant-mass limit, the parameter λ goes over to $\omega/2$, it is clear that in such a limit the quadratic energy spectrum (2.4) becomes linear and given by $E_{n,L} = \omega\left(2n + L + \frac{3}{2}\right)$. Furthermore, the mere definition of e, combined with limit relations between orthogonal polynomials [60] also allows us to retrieve the results for constant-mass wavefunctions $\psi_{n,L}(r)$, depending on the Laguerre polynomials [61].[1]

3. Spectrum generating algebra of the PDM *d*-dimensional radial harmonic oscillator

In order to build a counterpart of the su(1,1) spectrum generating algebra obtained in the constant-mass case [58], it is useful to start from a quadratic algebra approach. It has indeed been suggested [49, 50, 54] that for a whole class of Hamiltonians, such as those for which the bound-state wavefunctions can be written as the lowest-energy one multiplied by the increasing-degree polynomials in some variable t, there may exist an (in general nonlinear) algebra generating the spectrum, whose three generators are the Hamiltonian and the variable t, which are Hermitian operators, as well as their anti-Hermitian commutator. This algebra is characterized by a Casimir operator, which is some polynomial function of the three generators [49]. This is the approach to be followed in section 3.1.

3.1. Quadratic algebra approach to the spectrum generating algebra

Let us start from the Hamiltonian defined in equation (2.3), the variable t considered in (2.7) and their commutator,

$$\tilde{K}_1^{(\alpha)} = \pi_r^2 + \frac{L(L+1)}{r^2} + \frac{1}{4}\omega^2 r^2, \qquad \tilde{K}_2^{(\alpha)} = t, \qquad \tilde{K}_3^{(\alpha)} = -4\mathrm{i}\alpha\left(2\frac{r}{f}\pi_r + \mathrm{i}t\right). \tag{3.1}$$

From the basic commutator $[r, \pi_r] = \mathrm{i}f(\alpha; r)$, it is straightforward to derive the relations

$$\begin{aligned}
&\left[\tilde{K}_1^{(\alpha)}, \tilde{K}_2^{(\alpha)}\right] = \tilde{K}_3^{(\alpha)}, \\
&\left[\tilde{K}_2^{(\alpha)}, \tilde{K}_3^{(\alpha)}\right] = 8\alpha\left(1 - \tilde{K}_2^{(\alpha)2}\right), \\
&\left[\tilde{K}_3^{(\alpha)}, \tilde{K}_1^{(\alpha)}\right] = -8\alpha\left\{\tilde{K}_1^{(\alpha)}, \tilde{K}_2^{(\alpha)}\right\} - 16\alpha^2\left[\frac{\lambda}{\alpha}\left(\frac{\lambda}{\alpha} - 1\right) + L(L+1) - 1\right]\tilde{K}_2^{(\alpha)} \\
&\qquad - 16\alpha^2\left[\frac{\lambda}{\alpha}\left(\frac{\lambda}{\alpha} - 1\right) - L(L+1)\right],
\end{aligned} \tag{3.2}$$

showing that the operators $\tilde{K}_1^{(\alpha)}$, $\tilde{K}_2^{(\alpha)}$ and $\tilde{K}_3^{(\alpha)}$ generate a quadratic algebra. Its nature can be determined by comparing (3.2) with equation (3.2) of [49], defining the (general) Askey–Wilson algebra QAW(3) in terms of eight parameters R, A_1, A_2, C_1, C_2, D, G_1 and G_2. Since in the present case, $R = A_1 = C_1 = 0$, we have to deal here with a quadratic Jacobi algebra QJ(3), characterized by the parameters

$$\begin{aligned}
&A_2 = -8\alpha, \qquad C_2 = -16\alpha^2\left[\frac{\lambda}{\alpha}\left(\frac{\lambda}{\alpha} - 1\right) + L(L+1) - 1\right], \qquad D = 0, \qquad G_1 = 8\alpha, \\
&G_2 = -16\alpha^2\left[\frac{\lambda}{\alpha}\left(\frac{\lambda}{\alpha} - 1\right) - L(L+1)\right].
\end{aligned} \tag{3.3}$$

[1] Note that we obtain a phase factor $(-1)^n$ not present in equation (28.5) of [61]. This phase factor is consistent with the positive matrix elements for the su(1,1) generators and with standard wavefunctions for the one-dimensional harmonic oscillator (see section 4).

As $D^2 - 4A_2G_1 \neq 0$, this algebra is a non-degenerate one, i.e., an algebra that cannot be reduced to a Lie algebra by a change of basis.

From equation (3.4) of [49], we get the corresponding Casimir operator in the form

$$Q^{(\alpha)} = -16\alpha \tilde{K}_2^{(\alpha)} \tilde{K}_1^{(\alpha)} \tilde{K}_2^{(\alpha)} + \tilde{K}_3^{(\alpha)2} - 16\alpha^2 \left[\frac{\lambda}{\alpha}\left(\frac{\lambda}{\alpha} - 1\right) + L(L+1) - 1\right] \tilde{K}_2^{(\alpha)2} + 16\alpha \tilde{K}_1^{(\alpha)} - 32\alpha^2 \left[\frac{\lambda}{\alpha}\left(\frac{\lambda}{\alpha} - 1\right) - L(L+1)\right] \tilde{K}_2^{(\alpha)}. \quad (3.4)$$

Its eigenvalue can be obtained by inserting the explicit expressions (3.1) in (3.4) and is given by

$$Q^{(\alpha)} = 16\alpha^2 \left[\frac{\lambda}{\alpha}\left(\frac{\lambda}{\alpha} - 1\right) + L(L+1) - 2\right]. \quad (3.5)$$

Our aim now consists in constructing a positive-discrete series unitary irreducible representation of this algebra spanned by the Hamiltonian eigenfunctions $\psi_{n,L}^{(\alpha)}(r), n = 0, 1, 2, \ldots,$ which will be a counterpart of the su(1,1) representation D_k^+ with $k = \frac{1}{2}\left(L + \frac{3}{2}\right)$, obtained in the constant-mass case [58].

From the general theory developed in [49, 54], we know that in a basis ψ_p wherein the Hamiltonian, i.e., the generator $\tilde{K}_1^{(\alpha)}$, is diagonal, the unitary irreducible representations of QJ(3) are given by

$$\begin{aligned}
\tilde{K}_1^{(\alpha)} \psi_p &= \lambda_p \psi_p, \\
\tilde{K}_2^{(\alpha)} \psi_p &= a_{p+1} \psi_{p+1} + a_p \psi_{p-1} + b_p \psi_p, \\
\tilde{K}_3^{(\alpha)} \psi_p &= g_{p+1} a_{p+1} \psi_{p+1} - g_p a_p \psi_{p-1},
\end{aligned}$$

where λ_p, a_p, b_p and g_p are some real constants, which can be expressed in terms of the defining parameters (3.3) and read

$$\begin{aligned}
\lambda_p &= \alpha \left[4p(p+1) - \frac{\lambda}{\alpha}\left(\frac{\lambda}{\alpha} - 1\right) - L(L+1) + 1\right], \\
a_p^2 &= [16p^2(2p-1)(2p+1)]^{-1} \left(2p - \frac{\lambda}{\alpha} + L + 1\right)\left(2p - \frac{\lambda}{\alpha} - L\right) \\
&\quad \times \left(2p + \frac{\lambda}{\alpha} - L - 1\right)\left(2p + \frac{\lambda}{\alpha} + L\right), \\
b_p &= -[4p(p+1)]^{-1} \left(\frac{\lambda}{\alpha} - L - 1\right)\left(\frac{\lambda}{\alpha} + L\right), \\
g_p &= 8\alpha p.
\end{aligned} \quad (3.6)$$

An infinite-dimensional representation of the positive-discrete series type $D_{p_0}^+$ is then characterized by the properties $a_{p_0}^2 = 0$ and $a_p^2 > 0$ if $p = p_0 + n, n = 1, 2, \ldots$. From the explicit value of a_p^2 given in (3.6), it is clear that, for generic values of λ/α and L, such conditions can be achieved in a single way, namely by assuming

$$p_0 = \frac{1}{2}\left(\frac{\lambda}{\alpha} + L\right). \quad (3.7)$$

From (3.6) and (3.7), it results that the eigenvalues λ_{p_0+n} of $\tilde{K}_1^{(\alpha)}$ in $D_{p_0}^+$ coincide with the energy eigenvalues (2.4), i.e., $\lambda_{p_0+n} = E_{n,L}^{(\alpha)}, n = 0, 1, 2, \ldots$.

Furthermore, if we reset $\psi_{p_0+n} \to \psi^{(\alpha)}_{n,L}$, $a_{p_0+n} \to a^{(\alpha)}_{n,L}$, $b_{p_0+n} \to b^{(\alpha)}_{n,L}$ and $g_{p_0+n} \to g^{(\alpha)}_{n,L}$, the action of the generators $\tilde{K}^{(\alpha)}_2$ and $\tilde{K}^{(\alpha)}_3$ on the basis functions can be recast in the form

$$\begin{aligned} \tilde{K}^{(\alpha)}_2 \psi^{(\alpha)}_{n,L} &= a^{(\alpha)}_{n+1,L}\psi^{(\alpha)}_{n+1,L} + a^{(\alpha)}_{n,L}\psi^{(\alpha)}_{n-1,L} + b^{(\alpha)}_{n,L}\psi^{(\alpha)}_{n,L}, \\ \tilde{K}^{(\alpha)}_3 \psi^{(\alpha)}_{n,L} &= g^{(\alpha)}_{n+1,L}a^{(\alpha)}_{n+1,L}\psi^{(\alpha)}_{n+1,L} - g^{(\alpha)}_{n,L}a^{(\alpha)}_{n,L}\psi^{(\alpha)}_{n-1,L}, \end{aligned} \tag{3.8}$$

where

$$\begin{aligned} a^{(\alpha)}_{n,L} &= \frac{\tau_n}{\frac{\lambda}{\alpha} + 2n + L}\left(\frac{n(2n+2L+1)\left(\frac{\lambda}{\alpha}+n+L\right)\left(2\frac{\lambda}{\alpha}+2n-1\right)}{\left(\frac{\lambda}{\alpha}+2n+L-1\right)\left(\frac{\lambda}{\alpha}+2n+L+1\right)}\right)^{1/2}, \\ b^{(\alpha)}_{n,L} &= -\frac{\left(\frac{\lambda}{\alpha}-L-1\right)\left(\frac{\lambda}{\alpha}+L\right)}{\left(\frac{\lambda}{\alpha}+2n+L\right)\left(\frac{\lambda}{\alpha}+2n+L+2\right)}, \\ g^{(\alpha)}_{n,L} &= 4\alpha\left(\frac{\lambda}{\alpha}+2n+L\right), \end{aligned} \tag{3.9}$$

and τ_n is a phase factor depending on the choice made for the relative phase of $\psi^{(\alpha)}_{n,L}$ and $\psi^{(\alpha)}_{n-1,L}$. The first equation in (3.8) can be reduced to the recursion relation for the Jacobi polynomials $P_n^{\left(\frac{\lambda}{\alpha}-\frac{1}{2},L+\frac{1}{2}\right)}(t)$ and with the choice made in (2.8) for the normalization coefficients, we find that $\tau_n = +1$.

We conclude that the solutions of the PDM Schrödinger equation (2.3) can be derived by only using the quadratic algebra generated by the operators (3.1). To obtain from the latter the generators of a deformed su(1,1) spectrum generating algebra (and consequently a simpler construction of wavefunctions), we shall need to build some ladder operators, generalizing the constant-mass ones. Before proceeding to such a derivation in section 3.3, it is worth considering the constant-mass limit of the quadratic algebra that we have just introduced.

3.2. Constant-mass limit of the quadratic algebra

Although appropriate for solving the Schrödinger equation (2.3), the basis $\left(\tilde{K}^{(\alpha)}_1, \tilde{K}^{(\alpha)}_2, \tilde{K}^{(\alpha)}_3\right)$ of our quadratic algebra is not convenient to determine its $\alpha \to 0$ limit because $\tilde{K}^{(\alpha)}_2$ goes over to the constant -1. To circumvent this difficulty, it is necessary to go from $\left(\tilde{K}^{(\alpha)}_1, \tilde{K}^{(\alpha)}_2, \tilde{K}^{(\alpha)}_3\right)$ to a new basis $\left(\bar{K}^{(\alpha)}_1, \bar{K}^{(\alpha)}_2, \bar{K}^{(\alpha)}_3\right)$.

Let us set

$$\begin{aligned} \bar{K}^{(\alpha)}_1 &= \tilde{K}^{(\alpha)}_1 = \pi_r^2 + \frac{L(L+1)}{r^2} + \frac{1}{4}\omega^2 r^2, \\ \bar{K}^{(\alpha)}_2 &= \frac{1}{\alpha}\left(1-\tilde{K}^{(\alpha)}_2\right)^{-1}\left(1+\tilde{K}^{(\alpha)}_2\right) = r^2, \\ \bar{K}^{(\alpha)}_3 &= \frac{1}{2\alpha}\left\{\left(1-\tilde{K}^{(\alpha)}_2\right)^{-1}, \tilde{K}^{(\alpha)}_3\right\} = -2(2\mathrm{i}r\pi_r + f). \end{aligned}$$

Observe that the inverse transformation reads

$$\begin{aligned} \tilde{K}^{(\alpha)}_1 &= \bar{K}^{(\alpha)}_1, \qquad \tilde{K}^{(\alpha)}_2 = \left(1+\alpha\bar{K}^{(\alpha)}_2\right)^{-1}\left(-1+\alpha\bar{K}^{(\alpha)}_2\right), \\ \tilde{K}^{(\alpha)}_3 &= \alpha\left\{\left(1+\alpha\bar{K}^{(\alpha)}_2\right)^{-1}, \bar{K}^{(\alpha)}_3\right\}. \end{aligned} \tag{3.10}$$

Either from the commutation relations (3.2) of the first basis generators or by direct computation, we obtain for the second basis the commutation relations

$$\begin{aligned} &\left[\bar{K}^{(\alpha)}_1, \bar{K}^{(\alpha)}_2\right] = \tfrac{1}{2}\left\{1+\alpha\bar{K}^{(\alpha)}_2, \bar{K}^{(\alpha)}_3\right\}, \\ &\left[\bar{K}^{(\alpha)}_2, \bar{K}^{(\alpha)}_3\right] = 8\bar{K}^{(\alpha)}_2\left(1+\alpha\bar{K}^{(\alpha)}_2\right), \end{aligned}$$

$$\left[\bar{K}_3^{(\alpha)}, \bar{K}_1^{(\alpha)}\right] = 4\left\{1 + \alpha \bar{K}_2^{(\alpha)}, \bar{K}_1^{(\alpha)}\right\} - 16\alpha^2 \frac{\lambda}{\alpha}\left(\frac{\lambda}{\alpha} - 1\right)\bar{K}_2^{(\alpha)}\left(1 + \alpha \bar{K}_2^{(\alpha)}\right) + 4\alpha\left(1 + \alpha \bar{K}_2^{(\alpha)}\right)\left(1 + 3\alpha \bar{K}_2^{(\alpha)}\right).$$

In the $\alpha \to 0$ limit, it is obvious that these relations become linear. It is then straightforward to show that the resulting operators $\bar{K}_i = \lim_{\alpha\to 0} \bar{K}_i^{(\alpha)}$, $i = 1, 2, 3$, are some linear combinations of su(1,1) generators K_0, K_+, K_-, with commutation relations $[K_0, K_\pm] = \pm K_\pm$ and $[K_+, K_-] = -2K_0$. The results read $\bar{K}_1 = 2\omega K_0$, $\bar{K}_2 = (2/\omega)(K_+ + K_- + 2K_0)$ and $\bar{K}_3 = 4(K_+ - K_-)$.

Finally, on performing transformation (3.10) on the right-hand side of (3.4), the quadratic algebra Casimir operator yields, after some calculations, the relation

$$\begin{aligned} Q^{(\alpha)} - 16\alpha^2\left[\frac{\lambda}{\alpha}\left(\frac{\lambda}{\alpha} - 1\right) + L(L+1) - 2\right] &= \left(1 + \alpha \bar{K}_2^{(\alpha)}\right)^{-1}\left\{4\alpha^2\left[\bar{K}_3^{(\alpha)2} - 16\alpha^2\frac{\lambda}{\alpha}\left(\frac{\lambda}{\alpha} - 1\right)\bar{K}_2^{(\alpha)2} + 8\left\{\bar{K}_1^{(\alpha)}, \bar{K}_2^{(\alpha)}\right\}\right.\right. \\ &\quad \left.\left. + 12 - 16L(L+1)\right] + 160\alpha^3\bar{K}_2^{(\alpha)} + 112\alpha^4\bar{K}_2^{(\alpha)2}\right\}\left(1 + \alpha\bar{K}_2^{(\alpha)}\right)^{-1}. \end{aligned} \tag{3.11}$$

From equation (3.5), it follows that the operator between curly brackets on the right-hand side of (3.11) vanishes. Since $\omega^2 = 4\alpha^2 \frac{\lambda}{\alpha}\left(\frac{\lambda}{\alpha} - 1\right)$, we observe a close similarity between the first few terms making up this operator and the expression of the su(1,1) Casimir operator $C = -K_+K_- + K_0(K_0 - 1)$ in terms of $\bar{K}_1, \bar{K}_2, \bar{K}_3$, namely $C = \left(\bar{K}_3^2 - 4\omega^2\bar{K}_2^2 + 8\{\bar{K}_1, \bar{K}_2\}\right)/64$. We conclude that the substitution of a PDM for a constant mass has the effect of changing the constant $C = \frac{1}{4}\left(L + \frac{3}{2}\right)\left(L - \frac{1}{2}\right)$ into a function of r,

$$\begin{aligned} \bar{C}^\alpha(r) &\equiv \frac{1}{64}\left[\bar{K}_3^{(\alpha)2} - 16\alpha^2\frac{\lambda}{\alpha}\left(\frac{\lambda}{\alpha} - 1\right)\bar{K}_2^{(\alpha)2} + 8\left\{\bar{K}_1^{(\alpha)}, \bar{K}_2^{(\alpha)}\right\}\right] \\ &= \frac{1}{16}[(2L+3)(2L-1) - 10\alpha r^2 - 7\alpha^2 r^4]. \end{aligned} \tag{3.12}$$

3.3. Deformed su(1,1) spectrum generating algebra

The purpose of this subsection is to construct a third basis $\left(K_0^{(\alpha)}, K_+^{(\alpha)}, K_-^{(\alpha)}\right)$ of our quadratic algebra, satisfying the following three properties:

(i) $K_0^{(\alpha)}$ is proportional to the Hamiltonian of the problem, while $K_+^{(\alpha)}$ (resp. $K_-^{(\alpha)}$) is a raising (resp. lowering) ladder operator, which means that, up to some multiplicative factor, it transforms $\psi_{n,L}^{(\alpha)}$ into $\psi_{n+1,L}^{(\alpha)}$ (resp. $\psi_{n-1,L}^{(\alpha)}$) for any $n \in \mathbb{N}$ (resp. $n \in \mathbb{N}^+$) with the additional condition that $K_-^{(\alpha)}$ annihilates $\psi_{0,L}^{(\alpha)}$.

(ii) The operators $K_0^{(\alpha)}, K_+^{(\alpha)}, K_-^{(\alpha)}$ satisfy the same Hermiticity properties as K_0, K_+, K_-, i.e., $K_0^{(\alpha)\dagger} = K_0^{(\alpha)}$ and $K_\pm^{(\alpha)\dagger} = K_\mp^{(\alpha)}$.

(iii) In the $\alpha \to 0$ limit, they go over to the su(1,1) generators K_0, K_+, K_-.

From the known action of $\tilde{K}_2^{(\alpha)}$ and $\tilde{K}_3^{(\alpha)}$ on $\psi_{n,L}^{(\alpha)}$, given in (3.8), we can construct some n-dependent ladder operators

$$A_{+,n}^{(\alpha)} = \tilde{K}_3^{(\alpha)} + g_{n,L}^{(\alpha)}\tilde{K}_2^{(\alpha)} - g_{n,L}^{(\alpha)}b_{n,L}^{(\alpha)}, \qquad A_{-,n}^{(\alpha)} = \tilde{K}_3^{(\alpha)} - g_{n+1,L}^{(\alpha)}\tilde{K}_2^{(\alpha)} + g_{n+1,L}^{(\alpha)}b_{n,L}^{(\alpha)}. \tag{3.13}$$

It is indeed easy to check that

$$A_{+,n}^{(\alpha)}\psi_{n,L}^{(\alpha)} = a_{n+1,L}^{(\alpha)}\left(g_{n,L}^{(\alpha)} + g_{n+1,L}^{(\alpha)}\right)\psi_{n+1,L}^{(\alpha)}, \qquad A_{-,n}^{(\alpha)}\psi_{n,L}^{(\alpha)} = -a_{n,L}^{(\alpha)}\left(g_{n,L}^{(\alpha)} + g_{n+1,L}^{(\alpha)}\right)\psi_{n-1,L}^{(\alpha)}.$$

In (3.13), the quantum number n can be expressed in terms of $E^{(\alpha)}_{n,L}$ by inverting equation (2.4) and choosing the non-negative root of the resulting quadratic equation. The result reads

$$n = \frac{1}{2}\left[-\left(\frac{\lambda}{\alpha} + L + 1\right) + \delta_n\right], \qquad \delta_n = \sqrt{\frac{E^{(\alpha)}_{n,L}}{\alpha} + \frac{\lambda}{\alpha}\left(\frac{\lambda}{\alpha} - 1\right) + L(L+1)}.$$

We can now eliminate the n-dependence from $A^{(\alpha)}_{\pm,n}$ by replacing $E^{(\alpha)}_{n,L}$ by the Hamiltonian $H = \tilde{K}^{(\alpha)}_1$. This leads to the operators

$$A^{(\alpha)}_{\pm} = \tilde{K}^{(\alpha)}_3 - 4\alpha\tilde{K}^{(\alpha)}_2(1 \mp \delta) + 4\alpha\frac{\left(\frac{\lambda}{\alpha} - L - 1\right)\left(\left(\frac{\lambda}{\alpha} + L\right)\right.}{1 \pm \delta}, \tag{3.14}$$

where

$$\delta = \sqrt{\frac{\tilde{K}^{(\alpha)}_1}{\alpha} + \frac{\lambda}{\alpha}\left(\frac{\lambda}{\alpha} - 1\right) + L(L+1)}. \tag{3.15}$$

Although such operators satisfy condition (i) referred to above, they do not fulfil the remaining two conditions.

We can get rid of this shortcoming by multiplying $A^{(\alpha)}_{\pm}$ by some appropriate functions $F^{(\alpha)}_{\pm}\left(\tilde{K}^{(\alpha)}_1\right)$ of the Hamiltonian. Since the latter are not uniquely determined by conditions (ii) and (iii), we may choose them in such a way that the action of $K^{(\alpha)}_{\pm}$ on $\psi^{(\alpha)}_{n,L}$ is the simplest possible. Let us therefore define

$$K^{(\alpha)}_{\pm} = \pm\frac{1}{16\lambda}A^{(\alpha)}_{\pm}(\delta \pm 1)\sqrt{\frac{\delta \pm 2}{\delta}} = \pm\frac{1}{16\lambda}(\delta \mp 1)\sqrt{\frac{\delta}{\delta \mp 2}}A^{(\alpha)}_{\pm}, \tag{3.16}$$

leading to the relations

$$\begin{aligned} K^{(\alpha)}_{+}\psi^{(\alpha)}_{n,L} &= \frac{\alpha}{\lambda}\left[(n+1)\left(n + L + \frac{3}{2}\right)\left(n + \frac{\lambda}{\alpha} + L + 1\right)\left(n + \frac{\lambda}{\alpha} + \frac{1}{2}\right)\right]^{1/2}\psi^{(\alpha)}_{n+1,L}, \\ K^{(\alpha)}_{-}\psi^{(\alpha)}_{n,L} &= \frac{\alpha}{\lambda}\left[n\left(n + L + \frac{1}{2}\right)\left(n + \frac{\lambda}{\alpha} + L\right)\left(n + \frac{\lambda}{\alpha} - \frac{1}{2}\right)\right]^{1/2}\psi^{(\alpha)}_{n-1,L}. \end{aligned} \tag{3.17}$$

In (3.16), the factors $\pm\sqrt{(\delta \pm 2)/\delta}$ (alternatively $\pm\sqrt{\delta/(\delta \mp 2)}$) are required by condition (ii) above, whereas the factors $(\delta \pm 1)$ (alternatively $(\delta \mp 1)$) are optional ones having a simplifying effect on the matrix elements contained in (3.17).

The definition of the third basis is finally completed by

$$K^{(\alpha)}_0 = \frac{1}{4\lambda}\tilde{K}^{(\alpha)}_1,$$

such that

$$K^{(\alpha)}_0\psi^{(\alpha)}_{n,L} = \frac{1}{4\lambda}E^{(\alpha)}_{n,L}\psi^{(\alpha)}_{n,L}. \tag{3.18}$$

In the $\alpha \to 0$ limit, equations (3.17) and (3.18) agree with the standard su(1,1) results $K_{\pm}\psi_{n,L}(r) = \left[\left(n + \frac{1}{2} \pm \frac{1}{2}\right)\left(n + L + 1 \pm \frac{1}{2}\right)\right]^{1/2}\psi_{n\pm1,L}(r)$ and $K_0\psi_{n,L}(r) = (E_{n,L}/2\omega)\psi_{n,L}(r)$, respectively.

The three deformed su(1,1) generators $K^{(\alpha)}_0$, $K^{(\alpha)}_{+}$ and $K^{(\alpha)}_{-}$ satisfy the commutation relations

$$\begin{aligned} \left[K^{(\alpha)}_0, K^{(\alpha)}_{\pm}\right] &= \pm\frac{\alpha}{\lambda}K^{(\alpha)}_{\pm}(\delta \pm 1) = \pm\frac{\alpha}{\lambda}(\delta \mp 1)K^{(\alpha)}_{\pm}, \\ \left[K^{(\alpha)}_{+}, K^{(\alpha)}_{-}\right] &= -\frac{\alpha\delta}{\lambda}\left(2K^{(\alpha)}_0 + \frac{\alpha}{4\lambda}\right), \end{aligned}$$

which can be easily checked by applying both sides on any $\psi^{(\alpha)}_{n,L}$. Observe that for $\alpha \to 0$, we get $\alpha\delta/\lambda \to 1$ and $\alpha/\lambda \to 0$, so that the standard su(1,1) commutation relations are retrieved, as it should be.

The Casimir operator $C^{(\alpha)}$ of this deformed su(1,1) algebra can be written as $C^{(\alpha)} = -K^{(\alpha)}_+ K^{(\alpha)}_- + f\left(K^{(\alpha)}_0\right)$, where the function $f\left(K^{(\alpha)}_0\right)$ must be such that $C^{(\alpha)}$ commutes with $K^{(\alpha)}_+$ and that $f\left(K^{(\alpha)}_0\right) \to K_0(K_0 - 1)$ for $\alpha \to 0$. The latter condition of course determines $C^{(\alpha)}$ only up to some constant term of order $O(\alpha/\lambda)$. After some rather lengthy calculations, we arrive at the result

$$C^{(\alpha)} = -K^{(\alpha)}_+ K^{(\alpha)}_- + K^{(\alpha)2}_0 - \frac{\alpha}{\lambda}\left(\delta - \frac{5}{4}\right) K^{(\alpha)}_0 - \frac{\alpha^2}{8\lambda^2}\delta$$

leading to

$$C^{(\alpha)}\psi^{(\alpha)}_{n,L} = \left[\frac{1}{4}\left(1 - \frac{\alpha}{\lambda}\right)\left(L + \frac{3}{2}\right)\left(L - \frac{1}{2}\right) - \frac{3\alpha^2}{16\lambda^2} L(L+1)\right]\psi^{(\alpha)}_{n,L}. \tag{3.19}$$

Equation (3.19) should be contrasted with (3.12).

In the appendix, it is shown how the ladder operators $K^{(\alpha)}_+$ and $K^{(\alpha)}_-$ can be used to fully determine the functions $\psi^{(\alpha)}_{n,L}$ in a much more direct way than those sketched above equation (2.6) and below equation (3.9).

4. One-dimensional harmonic oscillator case

The purpose of this section is to show how the results of section 3, valid for $d \geqslant 2$, can be extended to the one-dimensional harmonic oscillator on the full line. This implies, in particular, replacing the radial variable r $(0 < r < \infty)$ by x $(-\infty < x < \infty)$.

For a constant mass, it is well known that apart from the substitution $r \to x$ the Schrödinger equation for the standard one-dimensional harmonic oscillator can be deduced from the d-dimensional radial one by setting either $L = -1$ or $L = 0$. In the former (resp. latter) case, one gets the even-parity (resp. odd-parity) wavefunctions and corresponding eigenvalues, $\psi_{\nu,-1}(r)/\sqrt{2} \to \psi_{2\nu}(x)$, $E_{\nu,-1} \to E_{2\nu}$ (resp. $\psi_{\nu,0}(r)/\sqrt{2} \to \psi_{2\nu+1}(x)$, $E_{\nu,0} \to E_{2\nu+1}$), due to some relations between the Laguerre and Hermite polynomials [60]. As a consequence, the single su(1,1) unitary irreducible representation D^+_k, $k = \frac{1}{2}\left(L + \frac{3}{2}\right)$, of the radial case gives rise to two such representations $D^+_{1/4}$ and $D^+_{3/4}$ (with the same Casimir $C = -3/16$), for the one-dimensional case.

The PDM Schrödinger equation

$$\left(\pi^2 + \tfrac{1}{4}\omega^2 x^2\right)\psi^{(\alpha)}(x) = E^{(\alpha)}\psi^{(\alpha)}(x),$$
$$\pi = \sqrt{f(\alpha; x)}\, p \sqrt{f(\alpha; x)}, \qquad p = -\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}x}, \qquad f(\alpha; x) = 1 + \alpha x^2,$$

equivalent to

$$\left(-\frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{M(x)}\frac{\mathrm{d}}{\mathrm{d}x} + V_{\mathrm{eff}}(x)\right)\psi^{(\alpha)}(x) = E^{(\alpha)}\psi^{(\alpha)}(x),$$
$$M(x) = \frac{1}{f^2(\alpha; x)} = \frac{1}{(1+\alpha x^2)^2}, \qquad V_{\mathrm{eff}}(x) = \frac{1}{4}(\omega^2 - 8\alpha^2)x^2 - \alpha,$$

admits a similar treatment exploiting the results obtained for equation (2.3), provided we distinguish again between the even- and odd-parity wavefunctions, given by

$$\psi^{(\alpha)}_{2\nu}(x) = \frac{\mathcal{N}^{(\alpha)}_{2\nu}}{\mathcal{N}^{(\alpha)}_0} P_\nu^{\left(\frac{\lambda}{\alpha} - \frac{1}{2}, -\frac{1}{2}\right)}(t)\,\psi^{(\alpha)}_0(x), \qquad \psi^{(\alpha)}_0(x) = \mathcal{N}^{(\alpha)}_0 f^{-(\lambda+\alpha)/(2\alpha)}$$

and

$$\psi^{(\alpha)}_{2\nu+1}(x) = \frac{\mathcal{N}^{(\alpha)}_{2\nu+1}}{\mathcal{N}^{(\alpha)}_1} P_\nu^{\left(\frac{\lambda}{\alpha}-\frac{1}{2},\frac{1}{2}\right)}(t)\psi^{(\alpha)}_1(x), \qquad \psi^{(\alpha)}_1(x) = \mathcal{N}^{(\alpha)}_1 x f^{-(\lambda+2\alpha)/(2\alpha)},$$

respectively. Here, $\nu = 0, 1, 2, \ldots, t = 1 - (2/f) = (-1+\alpha x^2)/(1+\alpha x^2)$,

$$\mathcal{N}^{(\alpha)}_0 = \left(\frac{\sqrt{\alpha}\Gamma\left(\frac{\lambda}{\alpha}+1\right)}{\sqrt{\pi}\Gamma\left(\frac{\lambda}{\alpha}+\frac{1}{2}\right)}\right)^{1/2}, \qquad \frac{\mathcal{N}^{(\alpha)}_{2\nu}}{\mathcal{N}^{(\alpha)}_0} = \left(\frac{\sqrt{\pi}\Gamma\left(\frac{\lambda}{\alpha}+\frac{1}{2}\right)\nu!\left(\frac{\lambda}{\alpha}+2\nu\right)\Gamma\left(\frac{\lambda}{\alpha}+\nu\right)}{\Gamma\left(\frac{\lambda}{\alpha}+1\right)\Gamma\left(\frac{\lambda}{\alpha}+\nu+\frac{1}{2}\right)\Gamma\left(\nu+\frac{1}{2}\right)}\right)^{1/2},$$

$$\mathcal{N}^{(\alpha)}_1 = \left(\frac{2\alpha^{3/2}\Gamma\left(\frac{\lambda}{\alpha}+2\right)}{\sqrt{\pi}\Gamma\left(\frac{\lambda}{\alpha}+\frac{1}{2}\right)}\right)^{1/2}, \qquad \frac{\mathcal{N}^{(\alpha)}_{2\nu+1}}{\mathcal{N}^{(\alpha)}_1} = \left(\frac{\sqrt{\pi}\Gamma\left(\frac{\lambda}{\alpha}+\frac{1}{2}\right)\nu!\left(\frac{\lambda}{\alpha}+2\nu+1\right)\Gamma\left(\frac{\lambda}{\alpha}+\nu+1\right)}{2\Gamma\left(\frac{\lambda}{\alpha}+2\right)\Gamma\left(\frac{\lambda}{\alpha}+\nu+\frac{1}{2}\right)\Gamma\left(\nu+\frac{3}{2}\right)}\right)^{1/2},$$

and the corresponding eigenvalues are

$$E^{(\alpha)}_n = \alpha\left(n^2 + (2n+1)\frac{\lambda}{\alpha}\right), \qquad \lambda = \frac{1}{2}(\alpha+\Delta), \qquad \Delta = \sqrt{\omega^2+\alpha^2}$$

in both cases $n = 2\nu$ and $n = 2\nu+1$.

There exists a quadratic spectrum generating algebra, for which we can construct three sets of generators $\left(\tilde{K}^{(\alpha)}_1, \tilde{K}^{(\alpha)}_2, \tilde{K}^{(\alpha)}_3\right)$, $\left(\bar{K}^{(\alpha)}_1, \bar{K}^{(\alpha)}_2, \bar{K}^{(\alpha)}_3\right)$ and $\left(K^{(\alpha)}_0, K^{(\alpha)}_+, K^{(\alpha)}_-\right)$, analogous to those built in section 3. The only differences lie in the substitutions $r \to x, \pi_r \to \pi$, $L(L+1) \to 0$, and in the very important fact that there are now two distinct unitary irreducible representations instead of a single one. This can be seen from the counterpart

$$a^2_p = [16p^2(2p-1)(2p+1)]^{-1}\left(2p - \frac{\lambda}{\alpha}\right)\left(2p - \frac{\lambda}{\alpha}+1\right)\left(2p+\frac{\lambda}{\alpha}\right)\left(2p+\frac{\lambda}{\alpha}-1\right)$$

of the similar quantity defined in (3.6). The conditions $a^2_{p_0} = 0$ and $a^2_p > 0$ if $p = p_0 + \nu, \nu = 1, 2, \ldots$, characterizing positive-discrete series representations $D^+_{p_0}$, are indeed satisfied now by two distinct values of p_0, $p_0 = \frac{1}{2}\left(\frac{\lambda}{\alpha}-1\right)$ and $p_0 = \frac{\lambda}{2\alpha}$, corresponding to $L = -1$ and $L = 0$ in (3.7) and to which we can associate $\lambda_{p_0+\nu} = E^{(\alpha)}_{2\nu}$ and $\lambda_{p_0+\nu} = E^{(\alpha)}_{2\nu+1}$, respectively.

Since, after these observations, it is straightforward to transpose the results of section 3 to the one-dimensional case, we are not going to detail them here. We would only like to mention that the action of the deformed su(1,1) generators on the wavefunctions reads

$$K^{(\alpha)}_0\psi^{(\alpha)}_n(x) = \frac{1}{4\lambda}E^{(\alpha)}_n\psi^{(\alpha)}_n(x) = \frac{\alpha}{4\lambda}\left(n^2+(2n+1)\frac{\lambda}{\alpha}\right)\psi^{(\alpha)}_n(x),$$

$$K^{(\alpha)}_+\psi^{(\alpha)}_n(x) = \frac{\alpha}{4\lambda}\left[(n+1)(n+2)\left(n+2\frac{\lambda}{\alpha}\right)\left(n+2\frac{\lambda}{\alpha}+1\right)\right]^{1/2}\psi^{(\alpha)}_{n+2}(x),$$

$$K^{(\alpha)}_-\psi^{(\alpha)}_n(x) = \frac{\alpha}{4\lambda}\left[n(n-1)\left(n+2\frac{\lambda}{\alpha}-2\right)\left(n+2\frac{\lambda}{\alpha}-1\right)\right]^{1/2}\psi^{(\alpha)}_{n-2}(x),$$

leading to the standard su(1,1) results $K_0\psi_n(x) = \frac{1}{2}\left(n+\frac{1}{2}\right)\psi_n(x)$, $K_\pm\psi_n(x) = \frac{1}{2}[(n\pm1)(n+1\pm1)]^{1/2}\psi_{n\pm2}(x)$ in the $\alpha \to 0$ limit.

5. Conclusion

In this paper, we have highlighted the interest of quadratic algebras for PDM Schrödinger equations by constructing spectrum generating algebras for a class of d-dimensional radial harmonic oscillators with $d \geqslant 2$ and a specific PDM choice, depending on some positive parameter α. We have also shown how minor changes enable the one-dimensional oscillator on the line with the same type of mass to be included in such a class.

For these quadratic algebras, we have considered three different sets of generators. The first one $\left(\tilde{K}_1^{(\alpha)}, \tilde{K}_2^{(\alpha)}, \tilde{K}_3^{(\alpha)}\right)$ has allowed us to prove the existence of a single unitary irreducible representation belonging to the positive-discrete series type for $d \geqslant 2$ and of two of them for $d = 1$, as well as to obtain the bound-state quadratic energy spectrum.

The second set $\left(\bar{K}_1^{(\alpha)}, \bar{K}_2^{(\alpha)}, \bar{K}_3^{(\alpha)}\right)$ has provided us with an explicit demonstration that the quadratic algebra considered here gives rise to the well-known su(1,1) Lie algebra generating the oscillator spectrum in the constant-mass limit, i.e., for $\alpha \to 0$.

This correspondence has been studied further by constructing a third set of operators $\left(K_0^{(\alpha)}, K_+^{(\alpha)}, K_-^{(\alpha)}\right)$, which go over to the standard su(1,1) generators (K_0, K_+, K_-) for $\alpha \to 0$ and may therefore be termed the deformed su(1,1) generators. All the bound-state wavefunctions have finally been built by using the lowering and raising generators, $K_-^{(\alpha)}$ and $K_+^{(\alpha)}$, respectively.

Some interesting open problems for future work are the extensions of the present study to other exactly solvable PDM Schrödinger equations either with the same potential but a different mass or with both different potential and mass.

Appendix

The purpose of this appendix is to prove equations (2.5)–(2.8) by using the deformed su(1,1) algebra introduced in section 3.3.

Let us start with $\psi_{0,L}^{(\alpha)}(r)$, which, according to the second relation in (3.17), is annihilated by $K_-^{(\alpha)}$ or, equivalently, by $A_-^{(\alpha)}$. Equations (3.14) and (3.15), together with (3.1), yield the first-order differential equation

$$r\frac{\mathrm{d}}{\mathrm{d}r}\psi_{0,L}^{(\alpha)}(r) = \left[-\frac{1}{2}\left(\frac{\lambda}{\alpha}+1\right)(1+t) + \frac{1}{2}(L+1)(1-t)\right]\psi_{0,L}^{(\alpha)}(r)$$

whose solution can be written in the form (2.5).

The excited-state wavefunctions $\psi_{n,L}^{(\alpha)}(r), n = 1, 2, \ldots,$ can now be determined recursively from $\psi_{0,L}^{(\alpha)}(r)$ by employing the first relation in (3.17). When combined with definition (3.16), the latter yields

$$\begin{aligned}\psi_{n+1,L}^{(\alpha)}(r) = {}& \frac{1}{16\alpha}\left(2n+\frac{\lambda}{\alpha}+L+2\right)\left(2n+\frac{\lambda}{\alpha}+L+3\right)^{1/2} \\ &\times\left[(n+1)\left(n+L+\frac{3}{2}\right)\left(n+\frac{\lambda}{\alpha}+L+1\right)\left(n+\frac{\lambda}{\alpha}+\frac{1}{2}\right)\right]^{-1/2} \\ &\times\left(2n+\frac{\lambda}{\alpha}+L+1\right)^{-1/2} A_+^{(\alpha)}\psi_{n,L}^{(\alpha)}(r). \end{aligned} \tag{A.1}$$

Let us now make the ansatz

$$\psi_{n,L}^{(\alpha)}(r) = \frac{\mathcal{N}_{n,L}^{(\alpha)}}{\mathcal{N}_{0,L}^{(\alpha)}}\psi_{0,L}^{(\alpha)}(r)P_n(t), \tag{A.2}$$

where $P_n(t)$ is some nth-degree polynomial in t, such that $P_0(t) = 1$. On inserting (A.2) in $A_+^{(\alpha)}\psi_{n,L}^{(\alpha)}(r)$ and using equations (3.1) and (3.14), we get

$$A_{+}^{(\alpha)}\psi_{n,L}^{(\alpha)}(r) = -8\alpha\frac{\mathcal{N}_{n,L}^{(\alpha)}}{\mathcal{N}_{0,L}^{(\alpha)}}\frac{\psi_{0,L}^{(\alpha)}(r)}{2n+\frac{\lambda}{\alpha}+L+2}\left\{\left(2n+\frac{\lambda}{\alpha}+L+2\right)(1-t^2)\frac{\mathrm{d}}{\mathrm{d}t}\right.$$
$$\left. -\left(n+\frac{\lambda}{\alpha}+L+1\right)\left[\frac{\lambda}{\alpha}-L-1+\left(2n+\frac{\lambda}{\alpha}+L+2\right)t\right]\right\}P_n(t)$$

which, according to (A.1) and (A.2), should be proportional to $\psi_{0,L}^{(\alpha)}(r)P_{n+1}(t)$. This clearly identifies $P_n(t)$ as the Jacobi polynomial $P_n^{(\beta,\gamma)}(t)$ with $\beta = \frac{\lambda}{\alpha}-\frac{1}{2}, \gamma = L+\frac{1}{2}$, because the latter satisfies the relation

$$\left\{(2n+\beta+\gamma+2)(1-t^2)\frac{\mathrm{d}}{\mathrm{d}t}-(n+\beta+\gamma+1)[\beta-\gamma+(2n+\beta+\gamma+2)t]\right\}$$
$$\times P_n^{(\beta,\gamma)}(t) = -2(n+1)(n+\beta+\gamma+1)P_{n+1}^{(\beta,\gamma)}(t) \tag{A.3}$$

obtained by eliminating $P_{n-1}^{(\beta,\gamma)}(t)$ between the Jacobi recursion and differential relations (see equations (22.7.1) and (22.8.1) of [60]). Hence equation (2.6) is proved.

Finally, on combining equations (A.1)–(A.3), we arrive at a recursion relation for the normalization coefficient

$$\frac{\mathcal{N}_{n+1,L}^{(\alpha)}}{\mathcal{N}_{n,L}^{(\alpha)}} = \left(\frac{(n+1)\left(n+\frac{\lambda}{\alpha}+L+1\right)\left(2n+\frac{\lambda}{\alpha}+L+3\right)}{\left(n+L+\frac{3}{2}\right)\left(n+\frac{\lambda}{\alpha}+\frac{1}{2}\right)\left(2n+\frac{\lambda}{\alpha}+L+1\right)}\right)^{1/2}$$

whose solution is given by (2.8). This completes the determination of the wavefunctions $\psi_{n,L}^{(\alpha)}(r)$.

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